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INVERSE OF A MATRIX AND ITS APPLICATIONS

Determinant of A Square Matrix

A square matrix A is said to be singular if its determinant is zero, i.e. $|A| = 0$

A square matrix A is said to be non-singular if its determinant is non-zero, i.e. $|A| \neq 0$

Minors and Cofactors of The Elements of Square Matrix

Minor

The Determinant that is left by cancelling the row and column intersecting at a particular element is called the minor of that element.

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ then Minor of a_{11} is

$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, Similarly $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

Using this concept the value of Determinant can be

$$\Delta = a_{11} M_{11} - a_{12} M_{12} + a_{13} M_{13}$$

Cofactor

The cofactor of an element a_{ij} is denoted by F_{ij} and is equal to $(-1)^{i+j} M_{ij}$ where M is a minor of element a_{ij}

if $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

then $F_{11} = (-1)^{1+1} M_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$F_{12} = (-1)^{1+2} M_{12} = -M_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

ADJOINT OF A SQUARE MATRIX

If every element of a square matrix A be replaced by its cofactor in |A|, then the transpose of the matrix so obtained is called the adjoint of matrix A and it is denoted by adj A

Thus if $A = [a_{ij}]$ be a square matrix and F^{ij} be the cofactor of a_{ij} in |A|, then

$$\text{Adj } A = [F^{ij}]^T$$

Hence if $A = \begin{bmatrix} a_{11} & a_{12} & \ldots a_{1n} \\ a_{21} & a_{22} & \ldots a_{2n} \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots a_{nn} \end{bmatrix}$, then

$$\text{Adj } A = \begin{bmatrix} F_{11} & F_{12} & \ldots F_{1n} \\ F_{21} & F_{22} & \ldots F_{2n} \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ F_{n1} & F_{n2} & \ldots F_{nn} \end{bmatrix}^T$$

INVERSE OF A MATRIX

If A and B are two matrices such that

$AB = I = BA$

then B is called the inverse of A and it is denoted by A^{-1}, thus

$A^{-1} = B \Leftrightarrow AB = I = BA$

To find inverse matrix of a given matrix A we use following formula

$$A^{-1} = \frac{\text{adj}A}{|A|}$$

Thus A^{-1} exists $\Leftrightarrow |A| \neq 0$

SOLUTION OF A SYSTEM OF LINEAR EQUATIONS

MATRIX METHOD

Let

$a_1x + b_1y = c_1$ ...(i)

$a_2x + b_2y = c_2$ ...(ii)

Matrix equation form

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

If A is singular, then $|A|=0$. Hence, A^{-1} does not exist

If not

$A^{-1}(AX) = A^{-1}B$

$(A^{-1}A)X = A^{-1}B$

$X = A^{-1}B$

CRITERION FOR CONSISTENCY OF A SYSTEM OF EQUATIONS

Let AX = B be a system of two or three linear equations.

1) If $|A| \neq 0$, then the system is consistent and has a unique solution, given by

$X = A^{-1}B$

2) If $|A| = 0$, then the system may or may not be consistent and if consistent, it does not have a unique solution. If in addition

(Adj A) B $\neq$ O then the system is inconsistent

(Adj A) B = O, then the system is consistent and has infinitely many solutions.

Check Your Progress

1. If cofactor of 2x in the determinant $\begin{vmatrix} x & 1 & -2 \\ 1 & 2x & x-1 \\ x-1 & x & 0 \end{vmatrix}$ is zero, then x equals to-

   (A) 0 (B) 2

   (C) 1 (D) –1

2. If $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and A_2, B_2, C_2 are respectively cofactors of a_2, b_2, c_2 then $a_1A_2 + b_1B_2 + c_1C_2$ is equal to-

   (A) $-\Delta$ (B) 0

   (C) Δ (D) None of these

3. The equations $x + 2y + 3z = 1$,

   $2x + y + 3z = 2$ and $5x + 5y + 9z = 4$ have-

   (A) unique solution (B) many solutions

(C) inconsistent (D) None of these

4. If $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$, then $|A + A^T|$ equals -

(A) $4(a^2 - b^2)$ (B) $2(a^2 - b^2)$

(C) $a^2 - b^2$ (D) 4 ab

5. For suitable matrices A, B; the false statement is-

(A) $(AB)^T = A^T B^T$

(B) $(A^T)^T = A$

(C) $(A - B)^T = A^T - B^T$

(D) $(A^T)^{-1} = (A^{-1})^T$

6. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then $A\,A'$ equals -

(A) I (B) A (C) A' (D) 0

7. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 0 & 2 \end{bmatrix}$, then the value of adj (adj A) is-

(A) $|A|^2$ (B) $-2A$

(C) 2A (D) A^2

8. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 1 \\ 2 & 1 & 2 \end{bmatrix}$, then A (adj A) equals-

(A) $\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ (B) $-\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 0 & 9 \\ 0 & 9 & 0 \\ 9 & 0 & 0 \end{bmatrix}$ (D) None of these

9. If $A = \begin{bmatrix} 1 & 2 \\ -4 & -1 \end{bmatrix}$, then $A^{-1} =$

(A) $\frac{1}{7}\begin{bmatrix} 1 & 2 \\ -4 & -1 \end{bmatrix}$ (B) $\begin{bmatrix} -1 & -2 \\ 4 & 1 \end{bmatrix}$

(C) $\frac{1}{9}\begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}$ (D) $\frac{1}{7}\begin{bmatrix} -1 & -2 \\ 4 & 1 \end{bmatrix}$

10. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then invertible matrices are-

(A) A and B (B) B and C

(C) A and C (D) All

Stretch Yourself

1. Find the value of x, y, z for the equation

$x + 2y + 3z = 1,\ 2x + y + 3z = 2,\ 5x + 5y + 9z = 4$

2. The system of linear equations $x + y + z = 2$, $2x + y - z = 3$, $3x + 2y + kz = 4$ has a unique solution .Find the value of k.

3. If α, β, γ are the roots of $x^3 + ax^2 + b = 0$, then Calculate the value of
$$\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$$

4. If $ax + by + cz = 1$, $bx + cy + az = 0 = cx + ay + bz$, then solve
$$\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix} \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

5. If matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & 6 & 7 \end{bmatrix}$ and its inverse is denoted by $A^{-1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then find the value of a_{23} .

Hint to Check Yourself

1C 2 B 3 A 4A 5 A

6 A 7 B 8B 9D 10C