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LIMIT AND CONTINUITY

If a function f (x) approaches L when x approaches 'a', we say that L is the limiting value of f (x) symbolically it is written as $\lim_{x\to 0} f(x) = L$

LEFT AND RIGHT HAND LIMITS

- If a function f (x) approaches a limit l_1, as x approaches 'a' from left, we say that the left hand limit of f(x) as x→a is l_1

$$\lim_{x\to a^-} f(x) = l_1$$

Or $\lim_{h\to 0} f(a-h) = l_1, h > 0$

- If a function f (x) approaches a limit l_2 , as x approaches 'a' fromright, we say that the right hand hand limit of f(x) as x→a is l_2

$$\lim_{x\to a^+} f(x) = l_2$$

Or $\lim_{h\to 0} f(a+h) = l_2, h > 0$

I. $\lim_{x\to a^+} f(x) = l$ and $\lim_{x\to a^-} f(x) = l$ ⇒
II. $\lim_{x\to a} f(x) = l$

III. $\lim_{x\to a^+} f(x) = l_1$ and $\lim_{x\to a^-} f(x) = l_2$ ⇒ $\lim_{x\to a} f(x)$not exit

IV. $\lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x)$ does not exit ⇒ $\lim_{x\to a} f(x)$not exit

BASIC THEOREMS ON LIMITS

I. $\lim_{x\to a} cx = c\lim_{x\to a} x$,c being a constant

II. $\lim_{x\to a}[g(x)+h(x)+p(x)+\cdots.] = \lim_{x\to a} g(x) + \lim_{x\to a} h(x) + \lim_{x\to a} p(x) + \ldots$..

III. $\lim_{x\to a}[f(x).g(x)] = \lim_{x\to a} f(x).\lim_{x\to a} g(x)$

IV. $\lim_{x\to a}\left\{\frac{f(x)}{g(x)}\right\} = \frac{\lim_{x\to a} x}{\lim_{x\to a} x}$

LIMITS OF SOME OF THE IMPORTANT FUNCTIONS

I. $\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}$ where n is a positive integer

II. $\lim_{x\to 0}\sin x = 0$ *and* $\lim_{x\to 0}\cos x = 1$

III. $\lim_{x\to 0}\frac{\sin x}{x} = 1$

IV. $\lim_{x\to 0}(1+x)^{1/x}$=e

V. $\lim_{x\to 0}\frac{\log(1+x)}{x} = \lim_{x\to 0}\frac{1}{x}\log(1+x) = \lim_{x\to 0}\log(1+x)^{1/x}$

VI. $\lim_{x\to 0}\frac{e^x-1}{x} = 1$

CONTINUITY OF A FUNCTION AT A POINT

1. A function f (x) is said to be continuous in an open inteval (a,b) if it is continuous at every point of (a,b).

2. A function f (x) is said to be continuous in the closed interval [a,b] if it is continuous at every point of the open interval]a,b[and is continuous at the point a from the right and continuous at b from the left.

$$\lim_{x \to a^+} f(x) = f(a)$$
and
$$\lim_{x \to b^-} f(x) = f(b)$$

Properties of Continuous Functions

If f (x) and g (x) are two functions which are continuous at a point x = a, then

(i) C f (x) is continuous at x = a, where C is a constant.
(ii) f (x) ±g(x) is continuous at x = a.
(iii) f (x). g(x) is continuous at x = a.)
(iv) f (x)/g (x) is continuous at x = a, provided $g \neq 0$
(v) |f(x)| is continuous at x = a.

Check Your Progress

1. If $f(x) = \begin{cases} 4x, & x < 0 \\ 1, & x = 0 \\ 3x^2, & x > 0 \end{cases}$, then $\lim_{x \to 0} f(x)$ equals-

(A) 0 (B) 1

(C) 3 (D) Does not exist

2. If $f(x) = \begin{cases} -1, & x < -1 \\ x^3, & -1 \le x \le 1 \\ 1-x, & 1 < x < 2 \\ 3-x^2, & x > 2 \end{cases}$ then-

(A) f(x) = 1 (B) $\lim_{x \to 1^+} f(x) = 1$

(C) $\lim_{x \to 2^+} f(x) = -1$ (D) $\lim_{x \to 2^-} f(x) = 0$

3. The value of $\lim_{x \to \pi/2} \frac{1-\sin^3 x}{\cos^2 x}$ is-

(A) $-\frac{3}{2}$ (B) $\frac{3}{2}$

(C) 1 (D) 0

4. The value of $\lim_{x \to 3} \left(\frac{x^4 - 81}{x-3} \right)$ is -

(A) –27 (B) 10

(C) undefined (D) None of these

5. $\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x^2} - \sqrt{1-x^2}}$ equals-

(A) 1 (B) 1/2

(C) 0 (D) Does not exist

6. $\lim_{x\to 3} \frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}$ equals-

(A) 0 (B) 3/2

(C) 1/4 (D) None of these

7. Function $f(x) = \begin{cases} 1+x, & \text{when } x < 2 \\ 5-x, & \text{when } x > 2 \end{cases}$; $x = 2$

is continuous at x = 2, if f(2) equals -

(A) 0 (B) 1

(C) 2 (D) 3

8. If $f(x) = \begin{cases} x\cos\frac{1}{x}, & x \neq 0 \\ k & x = 0 \end{cases}$ is continuous at x = 0, then

(A) k > 0 (B) k< 0

(C) k = 0 (D)k ≥ 0

9. If function $f(x) = \begin{cases} x^2+2, & x > 1 \\ 2x+1, & x = 1 \end{cases}$ is continuous at x = 1, then value of f(x) for x < 1 is-

(A) 3 (B) 1– 2x

(C) 1– 4x (D) None of these

10. If $f(x) = \begin{cases} \sin\frac{1}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$ is continuous at x = 0, then k is equal to -

(A) 8 (B) 1

(C) –1 (D) None of these

Stretch Yourself

1. If $f(x) = \frac{1-\cos(1-\cos x)}{x^4}$, (x ≠ 0) is continuous everywhere, then find f(0).
2. Is Function f(x) = $\begin{cases} \frac{(b^2-a^2)}{2}, & 0 \leq x \leq a \\ \frac{b^2}{2}-\frac{x^2}{6}-\frac{a^3}{3x}, & a < x \leq b, \\ \frac{1}{3}\left(\frac{b^3-a^3}{x}\right), & x > b \end{cases}$
3. If [x] denotes the greatest integer $\leq$ x, then

   Find $\lim_{n\to\infty} \frac{1}{n^3} \{[1^2 x] + [2^2 x] + [3^3 x] + \ldots + [n^2 x]$
4. Find the value of $\lim_{x\to\frac{\pi}{2}} \frac{\log\left(x-\frac{\pi}{2}\right)}{\tan x}$
5. Find $\lim_{x\to\pi/2} \frac{\cot x - \cos x}{(\pi-2x)^3}$

Hint to Check Yourself

1A	2 C	3 B	4 B	5 D
6 D	7 D	8 C	9 A	10 D