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# INTEGRATION

Integration is a reverse process of differentiation. The **integral or primitive** of a function f(x) with respect to x is that function (x) whose derivative with respect to x is the given function f(x). It is expressed symbolically as -

$$\int f(x)\,dx = \phi(x)$$

Thus

$$\int f(x)\,dx = \phi(x) \Leftrightarrow \frac{d}{dx}[\phi(x)] = f(x)$$

The process of finding the **integral** of a function is called Integration and the given function is called **Integrand**.

i. $\int 0 \,.\, dx = c$

ii. $\int 1 .dx = x + c$

iii. $\int k .dx = kx + c$

iv. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c\ (n–1)$

v. $\int \frac{1}{x}\,dx = \log_e x + c$

vi. $\int e^x\,dx = e^x + c$

vii. $\int a^x\,dx = \frac{a^x}{\log_e a} + c = a^x \log_a e + c$

viii. $\int \sin x\,dx = – \cos x + c$

ix. $\int \cos x\,dx = \sin x + c$

x. $\int \tan x\,dx = \log \sec x + c = – \log \cos x + c$

xi. $\int \cot x\,dx = \log \sin x + c$

xii. $\int \sec x\,dx = \log(\sec x + \tan x) + c$

$= – \log(\sec x – \tan x) + c$

$= \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c$

xiii. $\int \operatorname{cosec} x\,dx = – \log(\operatorname{cosec} x + \cot x) + c$

$= \log(\operatorname{cosec} x – \cot x) + c = \log \tan\left(\frac{x}{2}\right) + c$

xiv. $\int \sec x \tan x\,dx = \sec x + c$

xv. $\int \operatorname{cosec} x \cot x\,dx = – \operatorname{cosec} x + c$

xvi. $\int \sec^2 x\,dx = \tan x + c$

xvii. $\int \operatorname{cosec}^2 x\,dx = – \cot x + c$

xviii. $\int \sinh x\,dx = \cosh x + c$

xix. $\int \cosh x\,dx = \sinh x + c$

xx. $\int \operatorname{sech}^2 x\,dx = \tanh x + c$

xxi. $\int \operatorname{cosech}^2 x\,dx = – \coth x + c$

xxii. $\int \operatorname{sech} x \tanh x\,dx = – \operatorname{sech} x + c$

xxiii. $\int \operatorname{cosech} x \coth x = – \operatorname{cosech} x + c$

xxiv. $\int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$

xxv. $\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a} \log\left(\frac{x-a}{x+a}\right) + c$

xxvi. $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left(\frac{x+a}{x-a}\right) + c$

xxvii. $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

$= - \cos^{-1}\left(\frac{x}{a}\right) + c$

xxviii. $\int \frac{1}{\sqrt{x^2 + a^2}} dx = \sinh^{-1}\left(\frac{x}{a}\right) + c$

$= \log (x + \sqrt{x^2 + a^2}) + c$

xxix. $\int \frac{1}{\sqrt{x^2 - a^2}} dx = \cosh^{-1}\left(\frac{x}{a}\right) + c$

$= \log (x + \sqrt{x^2 - a^2}) + c$

xxx. $\int \sqrt{a^2 - x^2}\, dx$

$= \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} . \sin^{-1}\frac{x}{a} + c$

xxxi. $\int \sqrt{x^2 + a^2}\, dx$

$= \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} . \sinh^{-1}\frac{x}{a} + c$

xxxii. $\int \sqrt{x^2 - a^2}\, dx$

$= \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} . \cosh^{-1}\frac{x}{a} + c$

xxxiii. $\int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

xxxiv. $\int e^{ax} \sin bx\, dx$

$= \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) + c$

## INTEGRATION BY SUBSTITUTION:

When Integrand is a function of function -

i.e. $\int f\ [\varphi(x)]\ \varphi'(x)\, dx$

Here we put $\varphi(x) = t$ so that $\varphi'(x)\, dx = dt$ and in that case the integrand is reduced to $\int \mathbf{f(t)}\, dt$.

(ii) When integrand is the product of two factors such that one is the derivative of the other i.e. $I = \int \mathbf{f'(x)} f(x)\, dx$.

(iii) Integral of a function of the form $f(ax + b)$.

Here we put $ax + b = t$ and convert it into standard integral. Obviously if $\int \mathbf{f(x)\, dx} = \varphi(x)$, then

$\int \mathbf{f(ax+b)\, dx} = \mathbf{\frac{1}{a}}\, \varphi(ax + b)$

(iv) Standard form of Integrals:

(a) $\int \frac{f'(x)}{f(x)} dx = \log [f(x)] + c$

(b) $\int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + c$

(provided $n \neq -1$)

(c) $\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c$

## INTEGRATION BY PARTS :

**If u and v are two functions of x, then**

$$\int (u.v)\, dx = u\left(\int v\, dx\right) - \int\left(\frac{du}{dx}\right).\left(\int v\, dx\right)dx.$$

**If the integral is of the form $\int e^x$ [f(x) + f'(x)]dx,**

$$\int e^x[f(x)+f'(x)]dx = e^x f(x) + c$$

**If the integral is of the form $\int [x f'(x) + f(x)]dx$**

$$\int [x f'(x) + f(x)]dx = x f(x) + c$$

**When denominator can be factorized (using partial fraction) :**

Let the integrand is of the form $\frac{f(x)}{g(x)}$, where both f(x) and g(x) are polynomials. If degree of f(x) is greater than degree of g(x) then first divide f(x) by g(x) till the degree of the remainder becomes less than the degree of g(x).

$$\frac{f(x)}{g(x)} = Q(x) + \frac{R(x)}{g(x)}$$

**(i)** For every non repeated linear factor in the denominator, write

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

**(ii)** For repeated linear factors in the denominator, write-

$$\frac{1}{(x-a)^3(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3} + \frac{D}{(x-b)}$$

**(iii)** For every non repeated quadratic factor in the denominator, write

$$\frac{1}{(ax^2+bx+c)(x-d)} = \frac{Ax+B}{ax^2+bx+c} + \frac{C}{x-d}$$

## Check Yourself

1. $\int \sqrt{1+\sin 2x}\, dx$ equals-

   (A) sin x + cos x + c

   (B) sin x – cos x + c

   (C) cos x – sin x + c

   (D) None of these

2. $\int \frac{4+5\sin x}{\cos^2 x} dx$ equals-

   (A) 4 tan x – sec x + c
   (B) 4 tan x + 5 sec x + c
   (C) 9 tan x + c
   (D) None of these

3. $\int (\tan x + \cot x)\, dx$ equals-

   (A) log (tan x) + c
   (B) log (sin x + cosx) + c
   (C) log (cx)
   (D) None of these

4. $\int \frac{e^{5\log_e x} - e^{4\log_e x}}{e^{3\log_e x} - e^{2\log_e x}} dx$ equals-

   (A) $\frac{x^2}{2} + c$ (B) $\frac{x^3}{3} + c$

   (C) $\frac{x^4}{4} + c$ (D) None of these

5. $\int \frac{1-\cos 2x}{1+\cos 2x} dx$ equals-

(A) tan x + x + c (B) tan x – x + c

(C) sin x – x + c (D) sin x + x + c

6. The value of $\int \frac{\sin x + \cos x}{\sqrt{1+\sin 2x}}$ dx is-

(A) sin x + c (B) x + c

(C) cos x + c (D) $\frac{1}{2}$(sinx + cos x)

7. $\int a^{bx} b^{ax} dx$ is] where a, b □ $R^+$

(A) $\frac{a^{bx}b^{ax}}{\ell n(a^b b^a)} + c$

(B) $\frac{a^{bx}.\,b^{ax}}{\ell n\, a.\ell n\, b} + c$

(C) $\frac{a^{bx}.\,b^{ax}}{\ell n\, a^b.\ell n\, b^a} + c$

(D) None of these

8. $\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}$ dx equals-

(A) cot x + c (B) sec x + c

(C) tan x + c (D) cosec x + c

9. $\int \cos x\left(\frac{1}{\sin^2 x} + \frac{\sin x}{\cos^3 x}\right)$dx equals-

(A) sec x – cosec x + c

(B) cosec x – sec x + c

(C) sec x + cosec x + c

(D) None of these

10. The value of $\int \frac{dx}{(\sec^{-1} x)\, x\sqrt{x^2-1}}$ is-

(A) – log ($\sec^{-1}$ x) + c

(B) log ($\sec^{-1}$ x) + c

(C) $\frac{-(\sec^{-1} x)^{-2}}{2} + c$

(D) None of these

**11.** $\int \frac{3x^2}{x^6+1}$ dx equals-

(A) log ($x^6$ +1) + c

(B) $\tan^{-1}(x^3)$ + c

(C) 3 $\tan^{-1}(x^3)$ + c

(D) 3 $\tan^{-1}(x^3/3)$ + c

12. $\int \frac{\cos x}{1+\sin x}$ dx is equal to-

(A) – log (1+ sin x) + c

(B) log (1+ sin x) + c

(C) log (1– sin x) – c

(D) log (1– sin x) + c

13. $\int \cot x\, \mathrm{cosec}^2 x$ dx.

(A) $-\frac{1}{2}\cot^2 x + c$

(B) $\frac{1}{2}\cot^2 x + c$

(C) $-\frac{1}{2}\cos^2 x - c$

(D) None of these

14. $\int \frac{\log(\log x)}{x}$ dx equals-

(A) $\log x \log\left(\frac{\log x}{e}\right) + c$

(B) $\log(e/x^2) + c$

(C) $\log(x^2/e) + c$

(D) $\log x . \log(e/x) + c$

15. $\int x^3 e^{x^2} dx$ is equal to-

(A) $\frac{1}{2}(x^2 + 1)\, e^{x^2} + c$

(B) $\frac{1}{2}(x^2 - 1)\, e^{x^2} + c$

(C) $\frac{1}{2}(1 - x^2)\, e^{x^2} + c$

(D) None of these

16. $\int \frac{x - \sin x}{1 - \cos x}\, dx =$

(A) $x \cot\frac{x}{2} + c$ (B) $-x\cot\frac{x}{2} + c$

(C) $\cot\frac{x}{2} + c$ (D) None of these

17. $\int e^{2x}\left[\frac{1 + \sin 2x}{1 + \cos 2x}\right]dx$ -

(A) $\frac{1}{2}e^{2x}\cot x + c$

(B) $\frac{1}{2}e^{2x}\tan x + c$

(C) $-\frac{1}{2}e^{2x}\cot x + c$

(D) none

18. $\int \frac{dx}{x[(\log x)^2 + 4\log x - 1]} =$

(A) $\frac{1}{2\sqrt{5}}\log\left[\frac{\log x + 2 - \sqrt{5}}{\log x + 2 + \sqrt{5}}\right] + c$

(B) $\frac{1}{\sqrt{5}}\log\left[\frac{\log x + 2 - \sqrt{5}}{\log x + 2 + \sqrt{5}}\right] + c$

(C) $\frac{1}{2\sqrt{5}}\log\left[\frac{\log x + 2 + \sqrt{5}}{\log x + 2 - \sqrt{5}}\right] + c$

(D) $\frac{1}{\sqrt{5}}\log\left[\frac{\log x + 2 + \sqrt{5}}{\log x + 2 - \sqrt{5}}\right] + c$

19. $\int \frac{3x + 1}{2x^2 - 2x + 3}\, dx$ equals-

(A) $\frac{1}{4}\log(2x^2 - 2x + 3) - \frac{\sqrt{5}}{2}\tan^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + c$

(B) $\frac{3}{4}\log(2x^2 - 2x + 3) + \frac{\sqrt{5}}{2}\tan^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + c$

(C) $\frac{3}{4}\log(2x^2 - 2x + 3) + \frac{\sqrt{5}}{2}\tan^{-1}\left(\frac{4x-2}{5}\right) + c$

(D) None of these

20. $\int \frac{x^3 - x - 2}{(1-x^2)}\, dx =$

(A) $\log\frac{x+1}{x-1} - \frac{x}{2} + c$

(B) $\log\left(\frac{x-1}{x+1}\right) + \frac{x^2}{2} + c$

(C) $\log\left(\frac{x+1}{x-1}\right) + \frac{x^2}{2} + c$

(D) $\log\left(\frac{x-1}{x+1}\right) - \frac{x^2}{2} + c$

## Stretch Yourself

Find

1. $\int \frac{dx}{(x^2+1)(x^2+4)}$
2. $\int \frac{\cos x}{(1+\sin x)(2+\sin x)}\, dx$
3. $\int \frac{dx}{2x^2+x-1}$
4. $\int \frac{(\sec x\ \mathrm{cosec}\, x)}{\log\tan x}\, dx$
5. $\int \frac{x^{e-1} - e^{x-1}}{x^e - e^x}\, dx$
6. $\int \sec^4 x \tan x.\, dx$
7. $\int \frac{\cos x}{(1+\sin x)(2+\sin x)}\, dx$
8. $\int \frac{1+x^2}{\sqrt{1-x^2}}\, dx$

## Hint to Check Yourself

| | | | | |
|---|---|---|---|---|
| 1B | 2B | 3A | 4B | 5 B |
| 6B | 7A | 8C | 9A | 10B |
| 11B | 12B | 13A | 14A | 15B |
| 16B | 17B | 18A | 19B | 20D |