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Plane

A plane is a surface such that if any two points are taken on it, the line segment joining them lies completely on the surface. In other words, every point on the line segment joining any two points lies on the plane.

Equation of a plane passing through a given point

The general equation of a plane passing through a point (x_1, y_1, z_1) is

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$,

where a, b and c are constants.

Intercept form of a plane :

The equation of a plane intercepting lengths a, b and c with x- axis , y-axis and z-axis respectively is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Cartesian Form : If ℓ, m, n are direction cosines of the normal to a given plane which is at a distance p from the origin, then the equation of the plane is $\ell x + my + nz = p$.

A line perpendicular to a plane is called a normal to the plane. Clearly, every line lying in a plane is perpendicular to the normal to the plane.

For example : The direction ratios of a vector normal to the plane $3x + 2y + 5z - 6 = 0$ are 3, 2, 5 and hence a vector normal to the plane is $3\hat{i} + 2\hat{j} + 5\hat{k}$.

Vector equation of plane passing through a point and normal to a given vector

The vector equation of a plane passing through a point having position vector $\vec{n}$ is $(\vec{r} - \vec{a})\,\vec{n} = 0$

Reduction to cartesian form :

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

and $\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$

$(\vec{r} - \vec{a}) = (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}$

Then can be written as

$$\{(x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}\}.\{(a\hat{i} + b\hat{j} + c\hat{k})\} = 0$$

$$\Rightarrow a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

Thus, the coefficient of x, y, z in the cartesian equation of a plane are the direction ratios of normal to the plane.

Equation of plane in normal form

Vector form

The vector equation of a plane normal to unit vector $\hat{n}$ and at a distance d from the originis $\vec{r}.\hat{n} = d$

Cartesian form

If ℓ,, m, n, be the direction cosines of the normal to a given plane and p be the length of perpendicular from origin to the plane, then the equation of the plane is $\ell x + my + nz = p$.

(i) **Vector form** - The angle between the two planes is defined as the angle between normals.

Let θ be the angle between planes;

$\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ is given by

$$\cos\theta = \frac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

(ii) **Cartesian form** - The angle θ between the planes $a_1 x + b_1 y + c_1 z + d_1 = 0$ and

$a_2 x + b_2 y + c_2 z + d_2 = 0$ is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Intersection of plane

The equation of a plane passing through the intersection of $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is $(a_1x + b_1y + c_1z + d_1) + \lambda (a_2x + b_2y + c_2z + d_2) = 0$, where λ is a constant

Distance of a point from a plane

(i) **Vector form : -** The length of the perpendiclar from a point having position vector $\vec{a}$ to, the plane $\vec{r}.\vec{n} = d$ is given by $P = \frac{|\vec{a}.\vec{n} - d|}{|\vec{n}|}$

(ii) **Cartesian Form** : The length of the perpendicular from a point $P(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is given by

$$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Distance between the parallel planes

(i) **Vector form :** The distance between two parallel plane $\vec{r}.\vec{n} = d_1$ and $\vec{r}.\vec{n} = d_2$ is given by

$$d = \frac{|d_1 - d_2|}{|\vec{n}|}$$

(ii) **Cartesian form**

The distance between two parallel planes

$ax + by + cz + d_1 = 0$ and

$ax + by + cz + d_2 = 0$ is given by

$$d = \left|\frac{(d_2 - d_1)}{\sqrt{a^2 + b^2 + c^2}}\right|$$

Check Your Progress

1. If the line through the points (4, 1, 2) and (5, λ,, 0) is parallel to the line through the points (2, 1, 1) and (3, 3, –1), find λ,.

   (A) 3 (B) –3

   (C) 2 (D) 4

2. If co-ordinates of points P, Q, R, S are respectively (1, 2, 3), (4, 5, 7); (–4, 3, –6) and (2, 0, 2) then-

   (A) PQ ∥ RS (B) PQ ⊥ RS

   (C) PQ = RS (D) None of these

3. The point of intersection of lines $\frac{x-4}{5} = \frac{y-1}{2} = \frac{z}{1}$ and $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ is -

   (A) (–1, –1, –1) (B) (–1, –1, 1)

   (C) (1, –1, –1) (D) (–1, 1, –1)

4. The shortest distance between the lines

   $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$ is

   (A) $\sqrt{30}$ (B) $2\sqrt{30}$

   (C) $5\sqrt{30}$ (D) $3\sqrt{30}$

5. The equation of the plane through the three points (1, 1, 1), (1, –1, 1) and (–7, –3, –5), is-

   (A) 3x – 4z + 1 = 0

   (B) 3x – 4y + 1 = 0

   (C) 3x + 4y + 1 = 0

   (D) None of these

6. The co-ordinates of the foot of the perpendicular drawn from the origin to a plane is (2, 4, –3). The equation of the plane is-

   (A) 2x – 4y – 3z = 29

   (B) 2x – 4y + 3z = 29

   (C) 2x + 4y – 3z = 29

   (D) none of these

7. The equation of the plane through intersection of planes x + 2y + 3z = 4 and 2x + y – z = – 5 & perpendicular to the plane 5x + 3y + 6z + 8 = 0 is-

   (A) 7x – 2y + 3z + 81 = 0

   (B) 23x + 14y – 9z + 48 = 0

   (C) 51x + 15y – 50z + 173 = 0

   (D) None of these

8. The equation of the plane containing the line of intersection of the planes $2x - y = 0$ and $y - 3z = 0$ and perpendicular to the plane $4x + 5y - 3z - 8 = 0$ is-

(A) $28x - 17y + 9z = 0$

(B) $28x + 17y + 9z = 0$

(C) $28x - 17y - 9z = 0$

(D) $7x - 3y + z = 0$

9. Equations of the line through (1, 2, 3) and parallel to the plane $2x + 3y + z + 5 = 0$ are

(A) $\frac{x-1}{-1} = \frac{y-2}{1} = \frac{z-3}{-1}$

(B) $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{1}$

(C) $\frac{x-1}{3} = \frac{y-2}{2} = \frac{z-3}{1}$

(D) $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{1}$

10. The co-ordinates of the point where the line joining the points (2, –3, 1), (3, –4, –5) cuts the plane $2x + y + z = 7$ are-

(A) (2, 1, 0)   (B) (3, 2, 5)

(C) (1, –2, 7)   (D) None of these

11. The equation of the plane passing through the lines $\frac{x-4}{1} = \frac{y-3}{1} = \frac{z-2}{2}$ & $\frac{x-3}{1} = \frac{y-2}{-4} = \frac{z}{5}$ is-

(A) $11x - y - 3x = 35$

(B) $11x + y - 3z = 35$

(C) $11x - y + 3z = 35$

(D) none of these

12. The equation of the plane passing through the points (3, 2, 2) and (1, 0, –1) and parallel to the line $\frac{x-1}{2} = \frac{y-1}{-2} = \frac{z-2}{3}$, is-

(A) $4x - y - 2z + 6 = 0$

(B) $4x - y + 2z + 6 = 0$

(C) $4x - y - 2z - 6 = 0$

(D) none of these

13. The point where the line $\frac{x-1}{2}=\frac{y-2}{-3}=\frac{z+3}{4}$ meets the plane $2x + 4y - z = 1$, is-

(A) (3, –1, 1) (B) (3, 1, 1)

(C) (1, 1, 3) (D) (1, 3, 1)

14. The line drawn from (4, –1, 2) to the point (–3, 2, 3) meets a plane at right angles at the point (–10, 5, 4), then the equation of plane is-

(A) $7x - 3y - z + 89 = 0$

(B) $7x + 3y + z + 89 = 0$

(C) $7x - 3y + z + 89 = 0$

(D) none of these

15. The line $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ is parallel to the plane-

(A) $2x + 3y + 4z = 29$ (B) $3x + 4y - 5z = 10$

(C) $3x + 4y + 5z = 38$ (D) $x + y + z = 0$

Stretch Yourself

1. Find the distance between the line $\frac{x-1}{3}=\frac{y+2}{-2}=\frac{z-1}{2}$ & the plane $2x + 2y - z = 6$

2. Find the angle between the line $\frac{x-2}{a}=\frac{y-2}{b}=\frac{z-2}{c}$ and the plane $ax + by + cz + 6 = 0$

3. Find the angle between the line $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z+3}{-2}$ and the plane $x + y + 4 = 0$,

4. Find the equation of the plane containing the line $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$ and the point (0, 7, – 7)

5. Find the points on the line $\frac{x+1}{1}=\frac{y+3}{3}=\frac{z-2}{-2}$ distant $\sqrt{(14)}$ from the point in which the line meets the plane $3x + 4y + 5z - 5 = 0$

Hint to Check Your Progress

1 A 2D 3 A 4D 5A

6C 7C 8A 9A 10 C

11D 12D 13A 14 A 15B