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Straight line

Cartesian equation of a line passing through a given point and given direction ratio

Cartesian equation of a straight line passing through a fixed point (x_1, y_1, z_1) and having direction ratios a, b, c is $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

(1) The parametric equations of the line $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ are $x = x_1 + a\lambda$, $y = y_1 + b\lambda$, $z = z_1 + c\lambda$, where λ is the parameter.

(2) The coordinates of any point on the line $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ are $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$, where $\lambda \in R$.

(3) Since the direction cosines of a line are also direction ratios, therefore equation of a line passing through (x_1, y_1, z_1) and having direction cosines ℓ, m, n is $\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$

(4) Since x, y and z- axis passes through the origin and have direction cosines 1, 0, 0; 0, 1, 0 and 0, 0, 1 respectively. Therefore their equations are x-axis : $\frac{x-0}{1} = \frac{y-0}{0} = \frac{z-0}{0}$ or y = 0 and z = 0

y- axis : $\frac{x-0}{0} = \frac{y-0}{1} = \frac{z-0}{0}$ or x = 0 and z = 0

z- axis : $\frac{x-0}{0} = \frac{y-0}{0} = \frac{z-0}{1}$ or x = 0 and y = 0

Cartesian Equation of a line Passing Through Two Given Points

The cartesian equation of a line passing through two given points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

Perpendicular distance

Perpendicular Distance of Cartesian Form : To find the perpendicular distance of a given point $(\alpha, \beta, \gamma)$from a given line

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

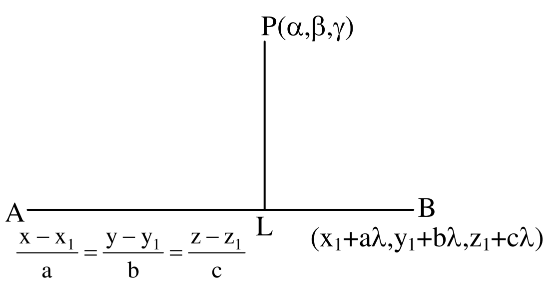
Let L be the foot of the perpendicular drawn from P (α, β, γ) on the line $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

Let the coordinates of L be $(x_1 + a\lambda,, y_1 + b\lambda,, z_1 + c\lambda,)$. Then direction ratios of PL are $x_1 + a\lambda, -\alpha,, y_1 + b\lambda, -\beta, z_1 + c\lambda, -\gamma$.

Direction ratio of AB are a, b, c. Since PL is perpendicular to AB, therefore

$$(x_1 + a\lambda - \alpha)\, a + (y_1 + b\lambda - \beta)\, b + (z_1 + c\lambda - \gamma)\, c = 0$$

$$\Rightarrow \lambda = \frac{a(\alpha - x_1) + b(\beta - y_1) + c(\gamma - z_1)}{a^2 + b^2 + c^2}$$



Putting this value of λin $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$, we obtain coordinates of L.

Now, using distance formula we can obtain the length PL

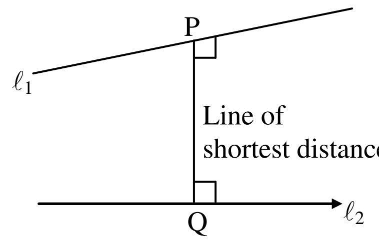
Skew Lines

Two straight lines in a space which are neither parallel nor intersecting are called skew-lines.

Thus, the skew lines are those lines which do not lie in the same plane.

(i) Shortest distance between two skew straight lines: If ℓ_1 and ℓ_2 are two skew lines, then there is one and only one line perpendicular to each of lines ℓ_1 and ℓ_2 which is known as three line of shortest distance.

Here, distance PQ is called to be shortest distance.



Vector form :

Let ℓ_1 and ℓ_2 be two lines whose equations are: $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$ respectively clearly ℓ_1 and ℓ_2 pass through the points A and B with position vectors $\vec{a}_1$ and $\vec{a}_2$ respectively and are parallel to the vectors $\vec{b}_1$ and $\vec{b}_2$ respectively

$$\text{Distance } \overrightarrow{PQ} = \left| \frac{(\vec{b}_1 \times \vec{b}_2).(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

Condition for lines to intersect

The two lines are intersecting if ;

$$\left| \frac{(\vec{b}_1 \times \vec{b}_2).(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right| = 0$$

$$\Rightarrow (\vec{b}_1 \times \vec{b}_2).(\vec{a}_2 - \vec{a}_1) = 0$$

□ $[\vec{b_1}\ \vec{b_2}\ (\vec{a_2}-\vec{a_1})] = 0$

Cartesian form :

Let the two skew lines be :

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$$

and $\dfrac{x-x_2}{a_2} = \dfrac{y-y_2}{b_2} = \dfrac{z-z_2}{c_2}$

shortest distance = $\left|\dfrac{(\vec{a_2}-\vec{a_1}).(\vec{b_1}\times\vec{b_2})}{|\vec{b_1}\times\vec{b_2}|}\right|$

d =

$$\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(m_1n_2-m_2n_1)^2+(n_1\ell_2-n_2\ell_1)^2+(\ell_1m_2-\ell_2m_1)^2}}$$

Conditions for lines to intersect

The lines are intersecting, if shortest distance = 0

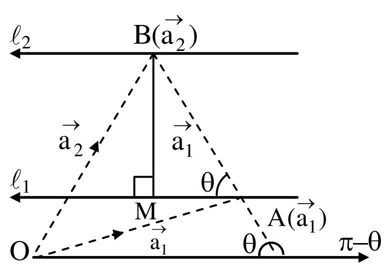
$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

(ii) Distance between parallel lines :

Let ℓ_1 and ℓ_2 are two parallel lines whose equations are

$\vec{r} = \vec{a_1} + \lambda\vec{b}$ and $\vec{r} = \vec{a_2} + \mu\vec{b}$

respetively



Clearly, ℓ_1 and ℓ_2 Pass through the points A and B with position vectors $\vec{a_1}$ and $\vec{a_2}$ respectively and both are parallel to the vector $\vec{b}$, where BM is the shortest distance between ℓ_1 and ℓ_2

shortest distance between parallel lines :

$\vec{r} = \vec{a_1} + \lambda\vec{b}$ and $\vec{r} = \vec{a_2} + \mu\vec{b}$ is :

$$d = \frac{|(\vec{a_2}-\vec{a_1})\times\vec{b}|}{|\vec{b}|}$$

Check Your Progress

1. If $\dfrac{x-1}{\ell} = \dfrac{y-2}{m} = \dfrac{z+1}{n}$ is the equation of the line through (1, 2, –1) & (–1, 0, 1), then (ℓ, m, n) is-

   (A) (–1, 0, 1)

   (B) (1, 1, –1)

   (C) (1, 2, –1)

   (D) (0, 1, 0)

2. If the angle between the lines whose direction ratios are 2, –1, 2 and a, 3, 5 be 45º, then a =

   (A) 1 (B) 2

   (C) 3 (D) 4

3. The co-ordinates of the foot of the perpendicular drawn from the point

A (1, 0, 3) to the join of the point B (4, 7, 1) and C (3, 5, 3) are-

(A) $\left(\frac{5}{3},\frac{7}{3},\frac{17}{3}\right)$ (B) (5, 7, 17)

(C) $\left(\frac{5}{3},-\frac{7}{3},\frac{17}{3}\right)$ (D) $\left(-\frac{5}{3},\frac{7}{3},-\frac{17}{3}\right)$

4. The length of the perpendicular from point(1, 2, 3) to the line $\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2}$ is-

(A) 5 (B) 6

(C) 7 (D) 8

5. The perpendicular distance of the point (2, 4, –1) from the line $\frac{x+5}{7} = \frac{y+3}{4} = \frac{z-6}{-9}$ is-

(A) 3 (B) 5

(C) 7 (D) none of these

6. The point of intersection of line $\frac{x-4}{5} = \frac{y-1}{2} = \frac{z}{1}$ and $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ is -

(A) (–1, –1, –1) (B) (–1, –1, 1)

(C) (1, –1, –1) (D) (–1, 1, –1)

7. The shortest distance between the lines

$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$ is

(A) $\sqrt{30}$ (B) $2\sqrt{30}$

(C) $5\sqrt{30}$ (D) $3\sqrt{30}$

8. The straight lines $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$ and $\frac{x-1}{2} = \frac{y-2}{2} = \frac{z-3}{-2}$ are-

(A) parallel lines

(B) intersecting at 60º

(C) skew lines

(D) intersecting at right angle

9. The equation of yz-plane is-

(A) x = 0 (B) y = 0

(C) z = 0 (D) x + y + z = 0

10. If the length of perpendicular drawn from origin on a plane is 7 units and its direction ratios are –3, 2, 6, then that plane is-

(A) –3x + 2y + 6z – 7 = 0

(B) –3x + 2y + 6z – 49 = 0

(C) 3x – 2y + 6z + 7 = 0

(D) –3x + 2y – 6z – 49 = 0

6A 7D 8D 9A 10B

Stretch Yourself

1. Find the foot of the perpendicular drawn from the point P (1, 0, 3) to the join of points A (4, 7, 1) and B (3, 5, 3) is –
2. Find the equation of the plane containing the line $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$ and the point (0, 7, – 7)
3. Find the distance of the point (1, – 2, 3) from the plane x – y + z = 5 measured parallel to the line $\frac{x}{2} = \frac{y}{3} = \frac{z-1}{-6}$
4. Find the points on the line $\frac{x+1}{1} = \frac{y+3}{3} = \frac{z-2}{-2}$ distant $\sqrt{(14)}$ from the point in which the line meets the plane 3x + 4y + 5z – 5 = 0
5. The lines $\frac{x-a+d}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{\alpha+\delta}$ and $\frac{x-b+c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b-c}{\beta+\gamma}$ are coplanar and then find the equation to the plane in which they lie,

Hint to Check Your progress

1 B 2 D 3 A 4C 5D