5

311en05

RELATIONS BETWEEN SIDES AND ANGLES OF A TRIANGLE

In earlier lesson, we have learnt about trigonometric functions of real numbers, relations between them, drawn the graphs of trigonometric functions, studied the characteristics from their graphs, studied about trigonometric functions of sum and difference of real numbers, and deduced trigonometric functions of multiple and sub-multiples of real numbers.

In this lesson, we shall try to establish some results which will give the relationship between sides and angles of a triangle and will help in finding unknown parts of a triangle.

OBJECTIVES

After studying this lesson, you will be able to :

- derive sine formula, cosine formula and projection formula
- apply these formulae to solve problems.

EXPECTED BACKGROUND KNOWLEDGE

- Trigonometric functions.
- Formulae for sum and difference of trigonometric functions of real numbers.
- Trigonometric functions of multiples and sub-multiples of real numbers.

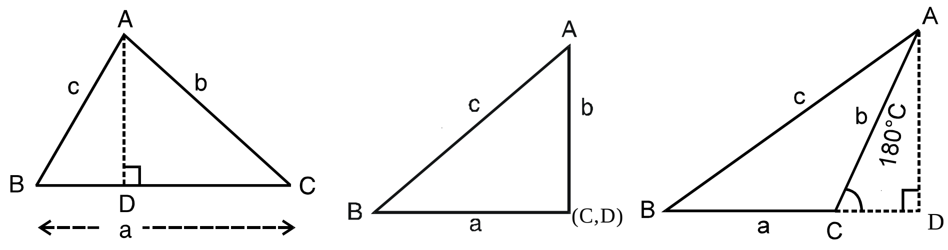
5.1 SINE FORMULA

In a ΔABC, the angles corresponding to the vertices A, B, and C are denoted by A, B, and C and the sides opposite to these vertices are denoted by a, b and c respectively. These angles and sides are called six elements of the triangle.

Prove that in any triangle, the lengths of the sides are proportional to the sines of the angles opposite to the sides,

i.e. $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Proof : In ΔABC, in Fig. 5.1 [(i), (ii) and (iii)], BC = a, CA = b and AB = c and $\angle C$ is acute angle in (i), right angle in (ii) and obtuse angle in (iii).



(i) (ii) (iii)

Fig. 5.1

Draw AD prependicular to BC (or BC produced, if need be)

In ΔABC, $\frac{AD}{AB} = \sin B$ or $\frac{AD}{c} = \sin B \Rightarrow AD = c \sin B$ .....(i)

In ΔADC, $\frac{AD}{AC} = \sin C$ in Fig 5.1 (i)

or, $\frac{AD}{b} = \sin C \Rightarrow AD = b \sin C$ ....(ii)

In Fig. 5.1 (ii), $\frac{AD}{AC} = 1 = \sin\frac{\pi}{2} = \sin C$ and $\frac{AD}{AB} = \sin B$

$AD = b \sin C$ and $AD = c \sin B$.

and in Fig. 5.1 (iii), $\frac{AD}{AC} = \sin(\pi - C) = \sin C$ and $\frac{AD}{AB} = \sin B$

or $\frac{AD}{b} = \sin C$ or $AD = b \sin C$ and $AD = c \sin B$

Thus, in all three figures, $AD = b \sin C$ and $AD = c \sin B$ ....(iii)

From (iii) we get

$$c \sin B = b \sin C \Rightarrow \frac{b}{\sin B} = \frac{c}{\sin C} \quad ....(iv)$$

Similarly, by drawing prependiculars from C on AB, we can prove that

$$\frac{a}{\sin A} = \frac{b}{\sin B} \quad .....(v)$$

From (iv) and (v), we get

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad .....(A)$$

(A) is called the sine-formula

Note : (A) is sometimes written as

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \qquad ....(A')$$

The relations (A) and (A') help us in finding unknown angles and sides, when some others are given.

Let us take some examples :

Example 5.1 Prove that $a \cos \frac{B-C}{2} = (b+c) \sin \frac{A}{2}$, using sine-formula.

Solution : We know that, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$ (say)

$$\Rightarrow \quad a = k \sin A,\ b = k \sin B,\ c = k \sin C$$

$$\therefore \quad \text{R.H.S.} = k(\sin B + \sin C) \cdot \sin \frac{A}{2}$$

$$= k \cdot 2 \sin \frac{B+C}{2} \cdot \cos \frac{B-C}{2} \cdot \sin \frac{A}{2}$$

Now $\quad \frac{B+C}{2} = 90° - \frac{A}{2} \qquad (\because \ A + B + C = \pi)$

$$\therefore \quad \sin \frac{B+C}{2} = \cos \frac{A}{2}$$

$$\therefore \quad \text{R.H.S.} = 2 k \cos \frac{A}{2} \cdot \cos \frac{B-C}{2} \cdot \sin \frac{A}{2}$$

$$= k \cdot \sin A \cdot \cos \frac{B-C}{2} = a \cdot \cos \frac{B-C}{2} = \text{L.H.S}$$

Example 5.2 Using sine formula, prove that

$$a(\cos C - \cos B) = 2(b-c)\cos^2 \frac{A}{2}$$

Solution : We have $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$ (say)

$$\Rightarrow \quad a = k \sin A,\ b = k \sin B,\ c = k \sin C$$

$$\therefore \quad \text{R.H.S} = 2k(\sin B - \sin C) \cdot \cos^2 \frac{A}{2}$$

$$= 2k \cdot 2 \cos \frac{B+C}{2} \cdot \sin \frac{B-C}{2} \cdot \cos^2 \frac{A}{2}$$

$$= 4k \sin\frac{A}{2}\cdot\sin\frac{B-C}{2}\cdot\ \cos^2\frac{A}{2} = 2a\sin\frac{B-C}{2}\cdot\ \cos\frac{A}{2}$$

$$= 2a\sin\frac{B+C}{2}\cdot\ \sin\frac{B-C}{2} \quad = a\left(\cos C - \cos B\right) = \text{L.H.S.}$$

Example 5.3 In any triangle ABC, show that

$$a\sin A - b\sin B = c\sin\left(A-B\right)$$

Solution : We have $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$ (say)

$$\text{L.H.S.} = k\sin A\cdot\sin A - k\sin B\cdot\sin B \ = k\left[\sin^2 A - \sin^2 B\right]$$

$$= k\sin\left(A+B\right)\cdot\sin\left(A-B\right)$$

$$A + B = \pi - C \Rightarrow \ \sin\left(A+B\right) = \sin C$$

$$\therefore \qquad \text{L.H.S.} = k\sin C\cdot\sin\left(A-B\right) \ = c\sin\left(A-B\right) = \text{R.H.S.}$$

Example 5.4 In any triangle, show that

$$a\left(b\cos C - c\cos B\right) = b^2 - c^2$$

Solution : We have, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$ (say)

$$\text{L.H.S.} = k\sin A\left(k\sin B\cos C - k\sin C\cos B\right) \ = k^2\cdot\sin A\left[\sin\left(B-C\right)\right]$$

$$= k^2\cdot\sin\left(B+C\right)\cdot\sin\left(B-C\right) \qquad \left[\because\ \sin A = \sin\left(B+C\right)\right]$$

$$= k^2\left(\sin^2 B - \sin^2 C\right) \ = k^2\sin^2 B - k^2\sin^2 C \ = b^2 - c^2 = \text{R.H.S}$$

CHECK YOUR PROGRESS 5.1

1. Using sine-formula, show that each of the following hold :

(i) $\dfrac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}} = \dfrac{a-b}{a+b}$ (ii) $b\cos B + c\cos C = a\cos\left(B-C\right)$

(iii) $a\sin\dfrac{B-C}{2} = \left(b-c\right)\cos\dfrac{A}{2}$ (iv) $\dfrac{b+c}{b-c} = \tan\dfrac{B+C}{2}\cdot\cot\ \dfrac{B-C}{2}$

(v) $a\cos A + b\cos B + c\cos C = 2a\sin B\sin C$

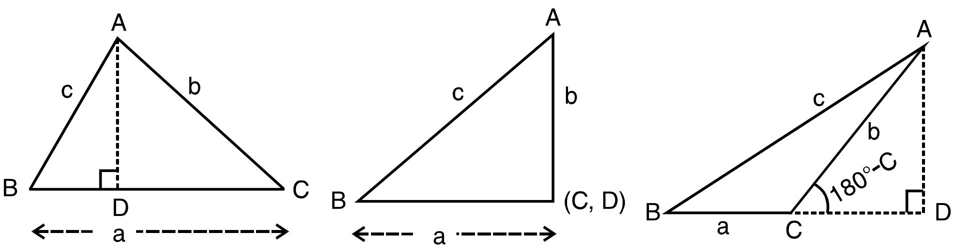
2. In any triangle if $\frac{a}{\cos A} = \frac{b}{\cos B}$, prove that the tiangle is isosceles.

5.2 COSINE FORMULA

In any triangle, prove that

(i) $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ (ii) $\cos B = \frac{c^2 + a^2 - b^2}{2ac}$ (iii) $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

Proof :



(i) (ii) (iii)

Fig. 5.2

Three cases arise :

(i) When $\angle C$ is acute (ii) When $\angle C$ is a right angle

(iii) When $\angle C$ is obtuse

Let us consider these one by one :

Case (i) When $\angle C$ is acute, $\frac{AD}{AC} = \sin C \quad \Rightarrow \quad AD = b \sin C$

Also $BD = BC - DC = a - b\cos C \quad \left[\because \frac{DC}{b} = \cos C\right]$

From Fig. 5.2 (i) $c^2 = (b\sin C)^2 + (a - b\cos C)^2$

$$= b^2 \sin^2 C + a^2 + b^2 \cos^2 C - 2ab\cos C \; = a^2 + b^2 - 2\,ab\,\cos C$$

$$\Rightarrow \quad \cos C = \frac{a^2 + b^2 - c^2}{2\,ab}$$

Case (ii) When $\angle C = 90°$, $c^2 = AD^2 + BD^2 = b^2 + a^2$

As $C = 90° \Rightarrow \cos C = 0 \therefore c^2 = b^2 + a^2 - 2ab \cdot \cos C$

$$\Rightarrow \quad \cos C = \frac{b^2 + a^2 - c^2}{2ab}$$

Case (iii) When $\angle C$ is obtuse

$$\frac{AD}{AC} = \sin(180° - C) = \sin C$$

$\therefore$ $AD = b \sin C$

Also, $BD = BC + CD = a + b\cos(180° - C)$

$= a - b\cos C \quad \therefore\ c^2 = (b\sin C)^2 + (a - b\cos C)^2$

$$= a^2 + b^2 - 2ab\cos C \Rightarrow \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$\therefore$ In all the three cases, $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Similarly, it can be proved that $\cos B = \dfrac{c^2 + a^2 - b^2}{2ac}$ and $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

Let us take some examples to show its application.

Example 5.5 In any triangle ABC, show that

$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$$

Solution : We know that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \ \cos B = \frac{c^2 + a^2 - b^2}{2ac}, \ \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\therefore \qquad \text{L.H.S.} = \frac{b^2 + c^2 - a^2}{2\,abc} + \frac{c^2 + a^2 - b^2}{2\,abc} + \frac{a^2 + b^2 - c^2}{2\,abc}$$

$$= \frac{1}{2\,abc}\left[b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2\right]$$

$$= \frac{a^2 + b^2 + c^2}{2\,abc} = \text{R.H.S.}$$

Example 5.6 If $\angle A = 60°$, show that in ΔABC

$$(a + b + c)(b + c - a) = 3bc$$

Solution : $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ .....(i) Here $A = 60° \Rightarrow \cos A = \cos 60° = \dfrac{1}{2}$

$\therefore$ (i) becomes $\frac{1}{2}=\frac{b^2+c^2-a^2}{2bc} \quad \Rightarrow \quad b^2+c^2-a^2=bc$

or $\quad b^2+c^2+2bc-a^2=3bc \quad$ or $\quad (b+c)^2-a^2=3bc$

or $\quad (b+c+a)(b+c-a)=3bc$

Example 5.7 If the sides of a triangle are 3 cm, 5 cm and 7 cm find the greatest angle of the triangle.

Solution : Here a = 3 cm, b = 5 cm, c = 7 cm

We know that in a triangle, the angle opposite to the largest side is greatest

$\therefore \quad \angle C$ is the greatest angle. $\therefore \quad \cos C=\frac{a^2+b^2-c^2}{2ab}$

$$=\frac{9+25-49}{30}=\frac{-15}{30}=\frac{-1}{2}$$

$\therefore \quad \cos C=\frac{-1}{2} \quad \Rightarrow \quad C=\frac{2\pi}{3}$

$\therefore$ The greatest angle of the triangle is $\frac{2\pi}{3}$ or $120°$.

Example 5.8 In ΔABC, if $\angle A=60°$, prove that $\frac{b}{c+a}+\frac{c}{a+b}=1$.

Solution : $\cos A=\frac{b^2+c^2-a^2}{2bc}$ or $\cos 60°=\frac{1}{2}=\frac{b^2+c^2-a^2}{2bc}$

$\therefore \quad b^2+c^2-a^2=bc \quad$ or $\quad b^2+c^2=a^2+bc \quad$ .....(i)

$$\text{L.H.S.}=\frac{b}{c+a}+\frac{c}{a+b}=\frac{ab+b^2+c^2+ac}{(c+a)(a+b)}$$

$$=\frac{ab+ac+a^2+bc}{(c+a)(a+b)} \quad \text{[Using (i)]}$$

$$=\frac{ab+a^2+ac+bc}{(c+a)(a+b)}=\frac{a(a+b)+c(a+b)}{(a+c)(a+b)}=\frac{(a+c)(a+b)}{(a+c)(a+b)}=1$$

CHECK YOUR PROGRESS 5.2

1. In any triangle ABC, show that

(i) $\frac{b^2-c^2}{a^2}\sin 2A+\frac{c^2-a^2}{b^2}\sin 2B+\frac{a^2-b^2}{c^2}\sin 2C=0$

(ii) $\left(a^2-b^2+c^2\right)\tan B=\left(b^2-c^2+a^2\right)\tan C=\left(c^2-a^2+b^2\right)\tan A$

(iii) $\frac{k}{2}(\sin 2A+\sin 2B+\sin 2C)=\frac{a^2+b^2+c^2}{2abc}$

where $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$

(iv) $\left(b^2-c^2\right)\cot A+\left(c^2-a^2\right)\cot B+\left(a^2-b^2\right)\cot C=0$

2. The sides of a triangle are $a = 9$ cm, $b = 8$ cm, $c = 4$ cm. Show that

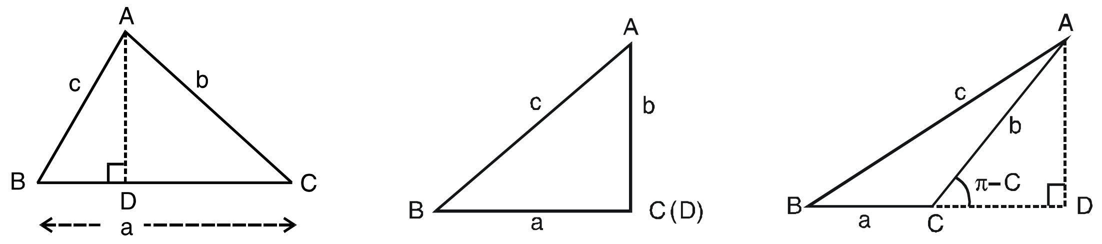
$$6\cos C = 4 + 3\cos B.$$

5.3 PROJECTION FORMULA

In ΔABC, if $BC = a$, $CA = b$ and $AB = c$, then prove that

(i) $a = b\cos C + c\cos B$ (ii) $b = c\cos A + a\cos C$ (iii) $c = a\cos B + b\cos A$

Proof :



(i) (ii) (iii)

Fig. 5.3

As in previous result, three cases arise. We will discuss them one by one.

(i) When $\angle C$ is acute :

In Δ ADB, $\frac{BD}{c}=\cos B$ $\Rightarrow$ $BD = c\cos B$

In Δ ADC, $\frac{DC}{b}=\cos C$ $\Rightarrow$ $DC = b\cos C$

$a = BD + DC = c\cos B + b\cos C$, $\therefore$ $a = c\cos B + b\cos c$

(ii) When $\angle C = 90°$

$$a = BC = \frac{BC}{AB} \cdot AB = \cos B \cdot c \quad = c \cos B + 0$$

$$= c \cos B + b \cos 90° \quad (\because \cos 90° = 0) \quad = c \cos B + b \cos C$$

(iii) When $\angle C$ is obtuse

In Δ ADB, $\frac{BD}{c} = \cos B \Rightarrow BD = c \cos B$

In Δ ADC, $\frac{CD}{b} = \cos(\pi - C) = -\cos C \quad \Rightarrow \quad CD = -b \cos C$

In Fig. 5.3 (iii),

$$BC = BD - CD \Rightarrow a = c \cos B - (-b \cos C) = c \cos B + b \cos C$$

Thus in all cases, $a = b \cos C + c \cos B$

Similarly, we can prove that

$$b = c \cos A + a \cos C \quad \text{and} \quad c = a \cos B + b \cos A$$

Let us take some examples, to show the application of these results.

Example 5.9 In any triangle ABC, show that

$$(b + c) \cos A + (c + a) \cos B + (a + b) \cos C = a + b + c$$

Solution : L.H.S. $= b \cos A + c \cos A + c \cos B + a \cos B + a \cos C + b \cos C$

$$= (b \cos A + a \cos B) + (c \cos A + a \cos C) + (c \cos B + b \cos C)$$

$$= c + b + a \quad = a + b + c = \text{R.H.S.}$$

Example 5.10 In any Δ ABC, prove that

$$\frac{\cos 2A}{a^2} - \frac{\cos 2B}{b^2} = \frac{1}{a^2} - \frac{1}{b^2}$$

Solution : L.H.S. $= \frac{1 - 2\sin^2 A}{a^2} - \frac{1 - 2\sin^2 B}{b^2}$

$$= \frac{1}{a^2} - \frac{2\sin^2 A}{a^2} - \frac{1}{b^2} + \frac{2\sin^2 B}{b^2}$$

$$= \frac{1}{a^2} - \frac{1}{b^2} - 2k^2 + 2k^2 = \frac{1}{a^2} - \frac{1}{b^2} \qquad \left(\therefore \frac{\sin A}{a} = \frac{\sin B}{b} = k\right)$$

= R. H. S.

Example 5.11 In Δ ABC, if $a \cos A = b \cos B$, where $a \neq b$ prove that Δ ABC is a right angled triangle.

Solution : $a\cos A = b\cos B$, $\therefore$ $a\left[\dfrac{b^2+c^2-a^2}{2bc}\right] = b\left[\dfrac{a^2+c^2-b^2}{2ac}\right]$

or $a^2\left(b^2+c^2-a^2\right) = b^2\left(a^2+c^2-b^2\right)$

or $a^2b^2+a^2c^2-a^4 = a^2b^2+b^2c^2-b^4$

or $c^2\left(a^2-b^2\right) = \left(a^2-b^2\right)\left(a^2+b^2\right)$

$\Rightarrow$ $c^2 = a^2+b^2$ $\therefore$ Δ ABC is a right triangle.

Example 5.12 If a = 2, b = 3, c = 4, find cos A, cos B and cos C.

Solution : $\cos A = \dfrac{b^2+c^2-a^2}{2bc} = \dfrac{9+16-4}{2\times3\times4} = \dfrac{21}{24} = \dfrac{7}{8}$

$\cos B = \dfrac{c^2+a^2-b^2}{2ac} = \dfrac{16+4-9}{2\times4\times2} = \dfrac{11}{16}$

and $\cos C = \dfrac{a^2+b^2-c^2}{2ab} = \dfrac{4+9-16}{2\times2\times3} = \dfrac{-3}{12} = \dfrac{-1}{4}$

CHECK YOUR PROGRESS 5.3

1. If a = 3, b = 4 and c = 5, find cos A, cos B and cos C.
2. The sides of a triangle are 7 cm, $4\sqrt{3}$ cm and $\sqrt{13}$ cm. Find the smallest angle of the triangle.
3. If a : b : c = 7 : 8 : 9, prove that
   cos A : cos B : cos C = 14 : 11 : 6.
4. If the sides of a triangle are x^2+x+1, $2x+1$ and x^2-1. Show that the greatest angle of the triangle is 120°.
5. In a triangle, b cos A = a cos B, prove that the triangle is isosceles.
6. Deduce sine formula from the projection formula.

LET US SUM UP

It is possible to find out the unknown elements of a triangle, if the relevent elements are given by using

Sine-formula :

(i) $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Cosine foumulae :

(ii) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, $\cos B = \dfrac{c^2 + a^2 - b^2}{2ac}$, $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Projection formulae :

$a = b\cos C + c\cos B$, $b = c\cos A + a\cos C$, $c = a\cos B + b\cos A$

SUPPORTIVE WEB SITES

www.mathopenref.com/trianglesideangle.html
http://en.wikipedia.org/wiki/Solution_of_triangles
www.themathpage.com/abookI/propI-18-19.htm

TERMINAL EXERCISE

In a triangle ABC, prove the following (1-10) :

1. $a\sin(B - C) + b\sin(C - A) + c\sin(A - B) = 0$

2. $a\cos A + b\cos B + c\cos C = 2a\sin B\sin C$

3. $\dfrac{b^2 - c^2}{a^2}\cdot\sin 2A + \dfrac{c^2 - a^2}{b^2}\cdot\sin 2B + \dfrac{a^2 - b^2}{c^2}\cdot\sin 2C = 0$

4. $\dfrac{c^2 + a^2}{b^2 + c^2} = \dfrac{1 + \cos B\cos(C - A)}{1 + \cos A\cos(B - C)}$

5. $\dfrac{c - b\cos A}{b - c\cos A} = \dfrac{\cos B}{\cos C}$

6. $\dfrac{a - b\cos C}{c - b\cos A} = \dfrac{\sin C}{\sin A}$

7. $(a + b + c)\left[\tan\dfrac{A}{2} + \tan\dfrac{B}{2}\right] = 2c\cot\dfrac{C}{2}$

8. $\sin\dfrac{A - B}{2} = \dfrac{a - b}{c}\cos\dfrac{C}{2}$

9. (i) $b\cos B + c\cos C = a\cos(B - C)$ (ii) $a\cos A + b\cos B = c\cos(A - B)$

10. $b^2 = (c - a)^2\cos^2\dfrac{B}{2} + (c + a)^2\sin^2\dfrac{B}{2}$

11. In a triangle, if $b = 5$, $c = 6$, $\tan\dfrac{A}{2} = \dfrac{1}{\sqrt{2}}$, then show that $a = \sqrt{41}$.

12. In any Δ ABC, show that $\dfrac{\cos A}{\cos B} = \dfrac{b - a\cos C}{a - b\cos C}$

ANSWERS

CHECK YOUR PROGRESS 5.3

1. $\cos A = \frac{4}{5}$

   $\cos B = \frac{3}{5}$

   $\cos C = \text{zero}$

2. The smallest angle of the triangle is 30°.