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# SOME SPECIAL SEQUENCES

Suppose you are asked to collect pebbles every day in such a way that on the first day if you collect one pebble, second day you collect double of the pebbles that you have collected on the first day, third day you collect double of the pebbles that you have collected on the second day, and so on. Then you write the number of pebbles collected daywise, you will have a sequence, $1, 2, 2^2, 2^3, .......$

From a sequence we derive a series. The series corresponding to the above sequence is

$1 + 2 + 2^2 + 2^3 + ......$

One well known series is Fibonacci series 1 + 1 + 2 + 3 + 5+ 8 + 13 + ....

In this lesson we shall study some special types of series in detail.

## OBJECTIVES

**After studying this lesson, you will be able to :**

- define a series;
- calculate the terms of a series for given values of *n* from $t_n$;
- evaluate $\sum n, \sum n^2, \sum n^3$ using method of differences and mathematical induction; and
- evaluate simple series like 1.3 + 3.5 + 5.7 +..... *n* terms.

## EXPECTED BACKGROUND KNOWLEDGE

- Concept of a sequence
- Concept of A. P. and G. P., sum of *n* terms.
- Knowldge of converting recurring decimals to fractions by using G. P.

## 7.1 SERIES

An expression of the form $u_1 + u_2 + u_3 + ... + u_n + ....$ is called a series, where $u_1, u_2, u_3 ..., u_n$ ... is a sequence of numbers. The above series is denoted by $\sum_{r=1}^{n} u_r$ . If $n$ is finite then the series is a finite series, otherwise the series is infinite. Thus we find that a series is associated to a sequence. Thus a series is a sum of terms arranged in order, according to some definite law.

Consider the following sets of numbers :

(a) 1, 6, 11, ...,  (b) $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12} \ldots$

(c) 48, 24, 12, ...,  (d) $1^2, 2^2, 3^2, \ldots.$

(a), (b), (c), (d) form sequences, since they are connected by a definite law. The series associated with them are :

$$1 + 6 + 11 + \ldots, \frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12} + \ldots, 48 + 24 + 12 + \ldots, 1^2 + 2^2 + 3^2 + \ldots$$

**Example 7.1** Write the first 6 terms of each of the following sequences, whose n^th^ term is given by

(a) $T_n = 2n + 1$,  (b) $a_n = n^2 - n + 1$  (c) $f_n = (-1)^n . 5^n$

Hence find the series associated to each of the above sequences.

**Solution :** (a) $T_n = 2n + 1$, For $n = 1$, $T_1 = 2.1 + 1 = 3$, For $n = 2$, $T_2 = 2.2 + 1 = 5$

For $n = 3$, $T_3 = 2.3 + 1 = 7$, For $n = 4$, $T_4 = 2.4 + 1 = 9$

For $n = 5$, $T_5 = 2.5 + 1 = 11$, For $n = 6$, $T_6 = 2.6 + 1 = 13$

Hence the series associated to the above sequence is $3 + 5 + 7 + 9 + 11 + 13 + \ldots$

(b) $a_n = n^2 - n + 1$, For $n = 1$, $a_1 = 1^2 - 1 + 1 = 1$

For $n = 2$, $a_2 = 2^2 - 2 + 1 = 3$, For $n = 3$, $a_3 = 3^2 - 3 + 1 = 7$

For $n = 4$, $a_4 = 4^2 - 4 + 1 = 13$,For $n = 5$, $a_5 = 5^2 - 5 + 1 = 21$

For $n = 6$, $a_6 = 6^2 - 6 + 1 = 31$

Hence the series associated to the above sequence is $1 + 3 + 7 + 13 + \ldots$

(c) Here $f_n = (-1)^n 5^n$, For $n = 1$, $f_1 = (-1)^1 5^1 = -5$

For $n = 2$, $f_2 = (-1)^2 5^2 = 25$, For $n = 3$, $f_3 = (-1)^3 5^3 = -125$

For $n = 4$, $f_4 = (-1)^4 5^4 = 625$, For $n = 5$, $f_5 = (-1)^5 5^5 = -3125$

For $n = 6$, $f_6 = (-1)^6 5^6 = 15625$

The corresponding series relative to the sequence

$f_n = (-1)^n 5^n$ is $-5 + 25 - 125 + 625 - 3125 + 15625 -$

**Example 7.2** Write the $n^{th}$ term of each of the following series :

(a) $-2 + 4 - 6 + 8 - ....$ (b) $1 - 1 + 1 - 1 + ....$

(c) $4 + 16 + 64 + 256 + ....$ (d) $\sqrt{2} + \sqrt{3} + 2 + \sqrt{5} + ....$

**Solution :** (a) The series is $-2 + 4 - 6 + 8 .......$

Here the odd terms are negative and the even terms are positive. The above series is obtained by multiplying the series. $-1 + 2 - 3 + 4 - ....$ by 2

$\therefore$ $T_n = 2(-1)^n n = (-1)^n 2n$

(b) The series is $1 - 1 + 1 - 1 + 1 - ......$

$\therefore$ $T_n = (-1)^{n+1}$

(c) The series is $4 + 16 + 64 + 256 + ....$

The above series can be writen as $4 + 4^2 + 4^3 + 4^4 + .....$

*i.e.*, $n^{th}$ term, $T_n = 4^n$.

(d) The series is $\sqrt{2} + \sqrt{3} + 2 + \sqrt{5} + ...$ *i.e.*, $\sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5} + ...$

$\therefore$ $n^{th}$ term is $T_n = \sqrt{n+1}$.

## CHECK YOUR PROGRESS 7.1

1. Write the first 6 terms of each of the following series, whose $n^{th}$ term is given by

(a) $T_n = \dfrac{n(n+1)(n+2)}{6}$ (b) $a_n = \dfrac{n^2 - 1}{2n - 3}$

2. If $A_1 = 1$ and $A_2 = 2$, find $A_6$ if $A_n = \dfrac{A_{n-1}}{A_{n-2}}, (n > 2)$

3. Write the $n^{th}$ term of each of the following series:

(a) $-1 + \dfrac{1}{2} - \dfrac{1}{3} + \dfrac{1}{4} - \cdots$ (b) $3 - 6 + 9 - 12 + ....$

## 7.2 SUM OF THE POWERS OF THE FIRST *n* NATURAL NUMBERS

(a) The series of first $n$ natural numbers is

$1 + 2 + 3 + 4 + .... + n$.

Let $S_n = 1 + 2 + 3 + ... + n$

This is an arithmetic series whose the first term is 1, the common difference is 1 and the number of terms is $n$. $\therefore$ $S_n = \frac{n}{2}\left[2.1+(n-1)1\right] = \frac{n}{2}\left[2n-1\right]$

*i.e.*, $S_n = \frac{n(n+1)}{2}$

$\therefore$ We can write $\sum n = \frac{n(n+1)}{2}$

(b) Determine the sum of the squares of the first $n$ natural numbers.

Let $S_n = 1^2 + 2^2 + 3^2 + .... + n^2$

Consider the identity : $n^3 - (n-1)^3 = 3n^2 - 3n + 1$

By giving the values for $n = 1, 2, 3, ...., n-1, n$ in the above identity, we have.

$1^3 - 0^3 = 3.1^2 - 3.1 + 1$

$2^3 - 1^3 = 3.2^2 - 3.2 + 1$

$3^3 - 2^3 = 3.3^2 - 3.3 + 1$

...... ......

...... ......

$n^3 - (n-1)^3 = 3n^2 - 3n + 1$

Adding these we get

$n^3 - 0^3 = 3\,(1^2 + 2^2 + 3^2 + ... + n^2) - 3\,(1 + 2 + 3 + ... + n) +$

$(1 + 1 + 1 + ...\ n$ times)

or, $n^3 = 3\,S_n - 3\left(\frac{n\,(n+1)}{2}\right) + n \ ....\left(\because \sum n = \frac{n\,(n+1)}{2}\right)$

or, $3\,S_n = n^3 + \frac{3n\,(n+1)}{2} - n = n\,(n^2 - 1) + \frac{3n}{2}(n+1)$

$= n\,(n+1)\left(n - 1 + \frac{3}{2}\right) = \frac{n\,(n+1)\,(2n+1)}{2}$

$\therefore$ $S_n = \frac{n\,(n+1)\,(2n+1)}{6}$ i.e., $\sum n^2 = \frac{n\,(n+1)\,(2n+1)}{6}$

(c) Determine the sum of the cubes of the first $n$ natural numbers.

Here $S_n = 1^3 + 2^3 + 3^3 + ... + n^3$

Consider the identity : $n^4 - (n-1)^4 = 4n^3 - 6n^2 + 4n - 1$

Putting successively 1, 2, 3, .... for $n$ we have

$$1^4 - 0^4 = 4.1^3 - 6.1^2 + 4.1 - 1$$

$$2^4 - 1^4 = 4.2^3 - 6.2^2 + 4.2 - 1$$

$$3^4 - 2^4 = 4.3^3 - 6.3^2 + 4.3 - 1$$

$$\ldots \quad \ldots$$

$$n^4 - (n-1)^4 = 4.n^3 - 6.n^2 + 4.n - 1$$

Adding these, we get

$$n^4 - 0^4 = 4(1^3 + 2^3 + \ldots + n^3) - 6(1^2 + 2^2 + \ldots + n^2) + 4\,(1 + 2 + 3 + \ldots + n) - (1 + 1 + \ldots\ n \text{ times})$$

$$\Rightarrow \qquad n^4 = 4.S_n - 6\left\{\frac{n\,(n+1)\,(2n+1)}{6}\right\} + 4n\,\frac{n+1}{2} - n$$

$$\Rightarrow \quad 4S_n = n^4 + n(n+1)\,(2n+1) - 2n\,(n+1) + n$$

$$= n^4 + n\,(2n^2 + 3n + 1) - 2n^2 - 2n + n$$

$$= n^4 + 2n^3 + 3n^2 + n - 2n^2 - 2n + n = n^4 + 2n^3 + n^2 = n^2\,(n^2 + 2n + 1)$$

*i.e.,* $4S_n = n^2\,(n+1)^2$

$$\therefore \qquad S_n = \frac{n^2\,(n+1)^2}{4} = \left\{\frac{n\,(n+1)}{2}\right\}^2$$

$$\therefore \quad \sum n^3 = \left[\frac{n\,(n+1)}{2}\right]^2 \quad \text{or,} \quad \sum n^3 = \left(\sum n\right)^2$$

> ***Note :*** *In problems on finding sum of the series, we shall find the nth term of the series ($t_n$) and then use $S_n = \sum t_n$.*

**Example 7.3** Find the sum of first $n$ terms of the series 1.3 + 3.5 + 5.7 + ...

**Solution :**

Let $S_n = 1.3 + 3.5 + 5.7 + \ldots$

The $n^{th}$ term of the series

$$t_n = \{n^{th} \text{ term of } 1, 3, 5, \ldots\} \times \{n^{th} \text{ term of } 3, 5, 7, \ldots\}$$

$$= (2n-1)\,(2n+1) = 4n^2 - 1$$

$$S_n = \sum t_n = \sum \left[4n^2 - 1\right]$$

$$= 4\sum n^2 - \sum(1) = 4\,\frac{n\,(n+1)\,(2n+1)}{6} - n$$

$$= \frac{2n\,(n+1)\,(2n+1) - 3n}{3} = \frac{n}{3}\left[2\,(2n^2 + 3n + 1) - 3\right]$$

$$= \frac{n}{3}\left[4n^2 + 6n - 1\right]$$

**Example 7.4** Find the sum of first $n$ terms of the series

$$1.2^2 + 2.3^2 + 3.4^2 + \ldots \ldots$$

**Solution :** Here $t_n = n\,\{2 + (n-1)\}^2 = n\,(n+1)^2 = n.\,(n^2 + 2n + 1)$

*i.e.,* $\quad t_n = n^3 + 2n^2 + n$

Let $\quad S_n = 1.2^2 + 2.3^2 + 2.3^2 + 3.4^2 + \ldots. + n.\,(n+1)^2.$

$\therefore \quad S_n = \sum t_n = \sum\,(n^3 + 2n^2 + n) = \sum n^3 + 2\sum n^2 + \sum n.$

$$= \left\{\frac{n\,(n+1)}{2}\right\}^2 + 2\left\{\frac{n\,(n+1)\,(2n+1)}{6}\right\} + \frac{n\,(n+1)}{2}$$

$$= n\,(n+1)\left\{\frac{n\,(n+1)}{4} + \frac{2n+1}{3} + \frac{1}{2}\right\}$$

$$= \frac{n\,(n+1)}{12}\,(3n^2 + 11n + 10) = \frac{1}{12}\,n\,(n+1)\,(n+2)\,(3n+5)$$

**Example 7.5** Find the sum of first $n$ terms of the series

$$2.\,3.\,5 + 3.\,5.\,7 + 4.\,7.\,9 + \ldots.$$

**Solution :** Let $S_n = 2.\,3.\,5. + 3.\,5.\,7 + 4.\,7.\,9 + \ldots.$

$n^{th}$ term of the series

$t_n =$ {$n^{th}$ term of 2, 3, 4, ...} × {$n^{th}$ term of 3, 5, 7, ...} × {$n^{th}$ term of 5, 7, 9, ....}

$= (n+1) \times (2n+1) \times (2n+3)$

$= (n+1)\,[4n^2 + 8n + 3] = 4n^3 + 12n^2 + 11n + 3$

$\therefore \quad S_n = \sum t_n = \sum\,[4n^3 + 12n^2 + 11n + 3]$

$= 4\sum n^3 + 12\sum n^2 + 11\sum n + \sum(3)$

$$= 4\frac{n^2(n+1)^2}{4} + \frac{12\,n\,(n+1)\,(2n+1)}{6} + \frac{11\,n\,(n+1)}{2} + 3n$$

$$= n^2(n+1)^2 + 2n\,(n+1)\,(2n+1) + \frac{11n\,(n+1)}{2} + 3n$$

$$= \frac{n}{2}\left[2n\,(n+1)^2 + 4\,(n+1)\,(2n+1) + 11\,(n+1) + 6\right]$$

$$= \frac{n}{2}\left[2n\,(n^2+2n+1) + 4\,(2n^2+3n+1) + 11n + 17\right]$$

$$= \frac{n}{2}\left[2n^3 + 12n^2 + 25n + 21\right]$$

**Example 7.6** Find the sum of first $n$ terms of the following series :

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots$$

**Solution :** $$t_n = \frac{1}{(2n-1)\,(2n+1)}$$

$$= \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

Now putting successively for $n = 1, 2, 3, \ldots.$

$$t_1 = \frac{1}{2}\left(1 - \frac{1}{3}\right)$$

$$t_2 = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right)$$

$$t_3 = \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right)$$

$$\ldots \qquad \ldots$$

$$t_n = \frac{1}{2}\left(\frac{1}{(2n-1)} - \frac{1}{(2n+1)}\right)$$

Adding, $$t_1 + t_2 + \cdots + t_n = \frac{1}{2}\left[1 - \frac{1}{2n+1}\right] = \frac{n}{(2n+1)}$$

## CHECK YOUR PROGRESS 7.2

1. Find the sum of first $n$ terms of each of the following series :

   (a) $1 + (1 + 3) + (1 + 3 + 5) + \ldots$

   (b) $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \cdots$

   (c) $(1) + (1 + 3) + (1 + 3 + 3^2) + (1 + 3 + 3^2 + 3^3) + \ldots$

2. Find the sum of $n$ terms of the series. whose $n^{th}$ term is $n(n + 1)(n + 4)$

3. Find the sum of the series $1. 2. 3 + 2. 3. 4. + 3. 4. 5 + \cdots$ upto n terms

## LET US SUM UP

- An expression of the form $u_1 + u_2 + u_3 + \ldots.. + u_n + \ldots$ is called a series, where $u_1, u_2, u_3 \ldots. u_n, \ldots.$ is a sequence of numbers

- $$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

- $$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

- $$\sum_{r=1}^{n} r^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

- $S_n = \sum t_n$

## SUPPORTIVE WEB SITES

http://en.wikipedia.org/wiki/Sequence_and_series
http://mathworld.wolfram.com/Series.html

## TERMINAL EXERCISE

1. Find the sum of each of the following series :

   (a) $2 + 4 + 6 + \ldots$ up to 40 terms.

(b) $2 + 6 + 18 + \ldots$ up to 6 terms.

2. Sum each of the following series to $n$ terms :

(a) $1 + 3 + 7 + 15 + 31 + \ldots..$

(b) $\dfrac{1}{1.3.5} + \dfrac{1}{3.5.7} + \dfrac{1}{5.7.9} + \ldots..$

(c) $\dfrac{3}{1.4} + \dfrac{5}{4.9} + \dfrac{7}{9.16} + \dfrac{9}{16.25} + \ldots.$

3. Find the sum of first $n$ terms of the series $1^2 + 3^2 + 5^2 + \ldots.$

4. Find the sum to $n$ terms of the series $5 + 7 + 13 + 31 + \ldots.$

5. Find the sum to $n$ terms of the series $1 + \dfrac{4}{5} + \dfrac{7}{5^2} + \dfrac{10}{5^3} + \cdots$

6. Find the sum of $2^2 + 4^2 + 6^2 + \ldots + (2n)^2$

7. Show that

$$\frac{1 \times 2^2 + 2 \times 3^2 + \ldots\ldots + n \times (n+1)^2}{1^2 \times 2 + 2^2 \times 3 + \ldots\ldots + n^2 \times (n+1)} = \frac{3n+5}{3n+1}$$

## ANSWERS

### CHECK YOUR PROGRESS 7.1

1. (a) 1, 4, 10, 20, 35, 56 (b) $0, 3, \frac{8}{3}, 3, \frac{24}{7}, \frac{35}{9}$

2. $\frac{1}{2}$

3. (a) $(-1)^n \frac{1}{n}$ (b) $(-1)^{n+1}\, 3n$

### CHECK YOUR PROGRESS 7.2

1. (a) $\frac{1}{6} n(n+1)(2n+1)$ (b) $\frac{n}{3n+1}$ (c) $\frac{1}{4}(3^{n+1}-2n-3)$

2. $\frac{n(n+1)}{12}\left[3n^2+23n+34\right]$

3. $\frac{1}{4} n(n+1)(n+2)(n+3)$

## TERMINAL EXERCISE

1. (a) 1640 (b) 728

2. (a) $2^{n+1}-n-2$ (b) $\frac{1}{12}-\frac{1}{4(2n+1)(2n+3)}$ (c) $1-\frac{1}{(n+1)^2}$

3. $\frac{n}{3}(4n^2-1)$

4. $\frac{1}{2}(3^n+8n-1)$

5. $\frac{5}{4}+\frac{15}{16}\left(1-\frac{1}{5^{n-1}}\right)-\frac{3n-2}{4.(5^{n-1})}$

6. $\frac{2n(n+1)(2n+1)}{3}$